Long Half-Life in Radioactive Decay

Mike Luciuk

The temperature of Earth's core is estimated to be ~7,000 K. As a result, average temperature increases by about 1° F for every 70' of depth. Most of this heat occurs from long lived radionuclides like uranium-238, thorium-232, and potassium-40. The numbers are the isotope atomic weights. The rest comes from the residual heat from Earth's formation. Although the total heat flows to Earth's surface averages about 0.1 watt/m^2, this is about 10,000 times less energy than we get from the Sun.

Since these isotopes are still present, and the Earth's age is about 4.6 billion years, it's obvious that they have significant lifetimes. A common measure of radionuclide lifetime is half-life, the time for half of the original nuclide to remain. These half- lives are in the range of billions of years.

Have you wondered how the half-lives of long-lived nuclides can be determined in relatively short periods of time? All we need are the atomic weight, sample mass, and number of decays in a short period of time. Since decay is a random process, we cannot predict when a particular atom will decay. However, the number of decays dN in a time frame of dt depends on the number of atoms in the sample, N. So $-\frac{dN}{dt} \propto N$. Nuclide decay rates vary, so each radionuclide has its own decay constant, λ. The reciprocal of the decay constant is the mean lifetime T of a particle before decay. Probability of decay, -dN/N, is proportional to the time increment dt.

Therefore,

$$-\frac{dN}{N} = \lambda dt \qquad \textbf{(1)}$$

The negative sign indicates that N decreases as time increases. The solution to this differential equation is

$$N = N_0 e^{-\lambda t} = N_0 e^{-t/T} \qquad \textbf{(2)}$$

where N_0 is the value of N at t=0. This can be rewritten as follows:

$$e^{-t/T} = \frac{N}{N_0} = \frac{N_0 - \Delta N}{N_0} \qquad \textbf{(3)}$$

where ΔN is the number of nuclides that have decayed in time t. We can use the Taylor series to expand $e^{-t/T}$.

$$e^{-t/T} = 1 + \frac{(-t/T)}{1!} + \frac{(-t/T)^2}{2!} + \frac{(-t/T)^3}{3!} + \dots\dots\dots \approx 1 - t/T \qquad \textbf{(4)}$$

because t is much smaller than T. Therefore, from equation (3)

$$1-{t}/{T}=\frac{N_0-\Delta N}{N_0}=1-{\Delta N}/{N_0} \quad \textbf{(5)}$$

resulting in
$$ {t}/{T}={\Delta N}/{N_0} \quad \textbf{(6)}$$

This is the relationship we can use to determine the half-life of a long lived radioactive nuclide.

Example: What is the half-life of U-238 if a 0.1 gram sample registers 1,240 becquerels (decays per second) on your Geiger counter?

Here, t = one second and ΔN = 1,240 decays. We need to find N_0, the original number of U-238 atoms in the 0.1 g sample. From high school chemistry, we know that one mole (238 g) of U-238 contains $6.022x10^{23}$ atoms, the well-known Avogadro's number. So one U-238 atom has a mass of ${238}/{6.022x10^{23}}$ = $3.95x10^{-22}$g. Therefore N_0, the number of U-238 atoms in our 0.1 g sample is ${0.1}/{3.95x10^{-22}}=2.53x10^{20}$ atoms. The average lifetime is

$$T=t\frac{N_0}{\Delta N}=(1)\frac{2.53x10^{20}}{1240}=2.04x10^{17}\text{seconds} \quad \textbf{(7)}$$

or 6.47 billion years (6.47Gyr).

To find half-life, we substitute $0.5N_0$ for N and 6.47 Gyr for T in equation (2)

$$0.5N_0=N_0e^{-{t}/{6.47Gyr}} \quad \textbf{(8)}$$

Solving for half-life, - t = ln(0.5)xT =- (0.693)x(6.47 Gyr) or $t_{half\text{-}life}$ = 4.48 billion years. Recall that ln is known as a natural logarithm, with base e.

References

http://en.wikipedia.org/wiki/Radioactive_decay
http://en.wikipedia.org/wiki/Geothermal_gradient